

ON THE BLOWUP FOR THE L^2 -CRITICAL FOCUSING NONLINEAR SCHRÖDINGER EQUATION IN HIGHER DIMENSIONS BELOW THE ENERGY CLASS

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ABSTRACT. We consider the focusing mass-critical nonlinear Schrödinger equation and prove that blowup solutions to this equation with initial data in $H^s(\mathbb{R}^d)$, $s > s_0(d)$ and $d \geq 3$, concentrate at least the mass of the ground state at the blowup time. This extends recent work by J. Colliander, S. Raynor, C. Sulem, and J. D. Wright, [13], T. Hmidi and S. Keraani, [21], and N. Tzirakis, [36], on the blowup of the two-dimensional and one-dimensional mass-critical focusing NLS below the energy space to all dimensions $d \geq 3$.

1. INTRODUCTION

We consider the initial value problem for the focusing L_x^2 -critical nonlinear Schrödinger equation

$$(1.1) \quad \begin{cases} iu_t + \Delta u = -|u|^{\frac{4}{d}}u \\ u(0, x) = u_0(x) \in H^s(\mathbb{R}^d), \quad s \geq 0, \end{cases}$$

where $u(t, x)$ is a complex-valued function in spacetime $\mathbb{R} \times \mathbb{R}^d$, $d \geq 3$.

It is well known (see, for example, [6]) that the Cauchy problem (1.1) is locally wellposed in $H^s(\mathbb{R}^d)$ for $s \geq 0$. Moreover, the unique solution obeys conservation of *mass*:

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 dx = M(u_0).$$

If $s \geq 1$, the *energy* is also finite and conserved:

$$E(u(t)) := \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u(t, x)|^2 - \frac{d}{2(d+2)} |u(t, x)|^{2+\frac{4}{d}} \right) dx = E(u_0).$$

This equation has a natural scaling. More precisely, the map

$$(1.2) \quad u(t, x) \mapsto u_\lambda(t, x) := \lambda^{\frac{d}{2}} u(\lambda^2 t, \lambda x)$$

maps a solution to (1.1) to another solution to (1.1). The reason why this equation is called L_x^2 -critical (or mass-critical) is because the scaling (1.2) also leaves the mass invariant.

Equation (1.1) is subcritical for $s > 0$. In this case, (1.1) is wellposed in $H^s(\mathbb{R}^d)$ and the lifespan of the local solution depends only on the H_x^s -norm of the initial data (see [6]). Denote by $T^* > 0$ the maximal forward time of existence. As

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a consequence of the local well-posedness theory, we have the following blowup criterion:

$$\text{either } T^* = \infty \quad \text{or} \quad T^* < \infty \text{ and } \lim_{t \rightarrow T^*} \|u(t)\|_{H_x^s} = \infty.$$

The blowup behavior for solutions from H_x^1 initial data has received a lot of attention. The results are closely related to the ground state Q which is the unique positive radial solution to the elliptic equation

$$\Delta Q - Q + |Q|^{\frac{4}{d}}Q = 0.$$

Using the sharp Gagliardo-Nirenberg inequality (see [39]),

$$(1.3) \quad \|u\|_{L_x^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_d \|u\|_{L_x^2}^{\frac{4}{d}} \|\nabla u\|_{L_x^2}^2 \quad \text{with} \quad C_d := \frac{d+2}{d \|Q\|_{L_x^2}^{\frac{4}{d}}},$$

it is not hard to see that the mass of the ground state is the minimal mass required for the solution to develop a singularity. Indeed, in the case $\|u_0\|_{L_x^2} < \|Q\|_{L_x^2}$, (1.3) combined with the conservation of energy imply that the solution to (1.1) is global. This is sharp since the pseudoconformal invariance of the equation (1.1) allows us to build a solution with mass equal to that of the ground state that blows up at time T^* :

$$u(t, x) := |T^* - t|^{-\frac{d}{2}} e^{[i(T^* - t)^{-1} - i|x|^2(T^* - t)^{-1}]} Q\left(\frac{x}{T^* - t}\right).$$

Moreover, F. Merle, [28], showed that up to the symmetries of (1.1), this is the only blowup solution with minimal mass. Furthermore, any blowup solution must concentrate at least the mass of the ground state at the blowup time; more precisely, as shown in [31], there exists $x(t) \in \mathbb{R}^d$ such that

$$(1.4) \quad \forall R > 0, \quad \lim_{t \rightarrow T^*} \int_{|x - x(t)| \leq R} |u(t, x)|^2 dx \geq \int_{\mathbb{R}^d} Q^2 dx.$$

Of course, the goal is to establish all these properties for blowup solutions from data in L_x^2 rather than H_x^1 . Unfortunately, all the methods used in the H_x^1 setting break down at the L_x^2 level. Moreover, as (1.1) is L_x^2 -critical, even the local well-posedness theory in L_x^2 is substantially different from that in H_x^s for $s > 0$. Specifically, the lifespan of the local solution depends on the profile of the initial data, rather than on its L_x^2 -norm (see [6]). In particular, this leads to the following blowup criterion:

$$\text{either } T^* = \infty \quad \text{or} \quad T^* < \infty \text{ and } \|u\|_{L_{t,x}^{2+\frac{4}{d}}([0, T^*) \times \mathbb{R}^d)} = \infty.$$

From the global theory for small data (see [6]), we know that if the mass of the initial data is sufficiently small, then there exists a unique global solution to (1.1). However, for large (but finite) mass initial data, that is also sufficiently smooth and decaying, the viriel identity guarantees that finite time blowup occurs; see, [18, 42].

The first blowup result for general L_x^2 initial data belongs to J. Bourgain, [2], who proved the following parabolic concentration of mass at the blowup time:

$$(1.5) \quad \lim_{t \nearrow T^*} \sup_{\substack{\text{cubes } I \subset \mathbb{R}^2 \\ \text{side}(I) < (T^* - t)^{\frac{1}{2}}}} \left(\int_I |u(t, x)|^2 dx \right)^{\frac{1}{2}} \geq c(\|u_0\|_{L_x^2}) > 0,$$

where $c(\|u_0\|_{L_x^2})$ is a small constant depending on the mass of the initial data. This result was extended to dimension $d = 1$ by S. Keraani, [25], and to dimensions $d \geq 3$ by P. Begout and A. Vargas, [1]. The conjecture is that rather than $c(\|u_0\|_{L_x^2})$ on the right-hand side of (1.5), one should have the mass of the ground state as in (1.4); however, this appears to be a very difficult problem.

The goal of this paper is to reproduce as much of the H_x^1 theory as we can at lower regularity (but unfortunately, well above L_x^2) in dimensions three and higher. In two dimensions, this was pursued by J. Colliander, S. Raynor, C. Sulem, and J. D. Wright, [13]; using the *I-method* (introduced by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, cf. [10]), they established a mass concentration property (at the blowup time) for radial blowup solutions from initial data in $H^s(\mathbb{R}^2)$ with $s > \frac{1+\sqrt{11}}{5}$:

$$(1.6) \quad \limsup_{t \nearrow T^*} \|u\|_{L^2(|x| < (T^* - t)^{\frac{s}{2}} \gamma(T^* - t))} \geq \|Q\|_{L_x^2},$$

where $\gamma(s) \rightarrow \infty$ arbitrarily slowly as $s \rightarrow 0$. While the right-hand side in (1.6) is the conjectured one, we should remark that the width of concentration, that is, $(T^* - t)^{\frac{s}{2}+}$ is larger than the expected concentration width, $(T^* - t)^{\frac{1}{2}+}$. Moreover, the limsup in (1.6) is not present in the H_x^1 theory (see (1.4)); the reason for its appearance is basically the lack of information on the blowup rate of the H_x^s -norm.

T. Himidi and S. Keraani, [21], removed the radial assumption used in [13]. Moreover, they showed that, up to symmetries of the equation, the ground state Q is the profile for blowup solutions with minimal mass and initial data in $H^s(\mathbb{R}^2)$ with $s > \frac{1+\sqrt{11}}{5}$. We will make use of the key innovation of their work, namely Lemma 2.11 below.

In one dimension, recent work by N. Tzirakis, [36], established the analogue of (1.6) for arbitrary initial data in $H^s(\mathbb{R})$, $s > \frac{10}{11}$.

Our contribution is to treat (non-radial) data in $H^s(\mathbb{R}^d)$ for $s > s_0(d)$ where

$$(1.7) \quad s_0(d) := \begin{cases} \frac{1+\sqrt{13}}{5}, & \text{for } d = 3, \\ \frac{8-d+\sqrt{9d^2+64d+64}}{2(d+10)}, & \text{for } d \geq 4. \end{cases}$$

We prove

Theorem 1.1. *Assume $d \geq 3$ and $s > s_0(d)$. Let $u_0 \in H^s(\mathbb{R}^d)$ such that the corresponding solution u to (1.1) blows up at time $0 < T^* < \infty$. Then, there exists a function $V \in H_x^1$ such that $\|V\|_2 \geq \|Q\|_2$ and there exist sequences $\{t_n, \rho_n, x_n\}_{n \geq 1} \subset \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}^d$ satisfying*

$$t_n \nearrow T^* \text{ as } n \rightarrow \infty \quad \text{and} \quad \rho_n \lesssim (T^* - t_n)^{\frac{s}{2}}, \quad \forall n \geq 1$$

such that

$$\rho_n^{\frac{d}{2}} u(t_n, \rho_n \cdot + x_n) \rightharpoonup V \quad \text{weakly} \quad \text{as } n \rightarrow \infty.$$

As a consequence of Theorem 1.1, we establish the following mass concentration property for blowup solutions:

Theorem 1.2. *Assume $d \geq 3$ and $s > s_0(d)$. Let $u_0 \in H^s(\mathbb{R}^d)$ such that the corresponding solution u to (1.1) blows up at time $0 < T^* < \infty$. Let $\alpha(t) > 0$ be such that*

$$\lim_{t \nearrow T^*} \frac{(T^* - t)^{\frac{s}{2}}}{\alpha(t)} = 0.$$

Then, there exists $x(t) \in \mathbb{R}^d$ such that

$$\limsup_{t \nearrow T^*} \int_{|x-x(t)| \leq \alpha(t)} |u(x, t)|^2 dx \geq \int_{\mathbb{R}^d} Q^2 dx.$$

Under the additional hypothesis that the mass of the initial data equals the mass of the ground state, we may upgrade Theorem 1.1 to the following:

Theorem 1.3. *Assume $d \geq 3$ and $s > s_0(d)$. Let $u_0 \in H^s(\mathbb{R}^d)$ with $\|u_0\|_2 = \|Q\|_2$ such that the corresponding solution u to (1.1) blows up at time $0 < T^* < \infty$. Then, there exist sequences $\{t_n, \theta_n, \rho_n, x_n\}_{n \geq 1} \subset \mathbb{R}_+ \times S^1 \times \mathbb{R}_+^* \times \mathbb{R}^d$ satisfying*

$$t_n \nearrow T^* \text{ as } n \rightarrow \infty \quad \text{and} \quad \rho_n \lesssim (T^* - t_n)^{\frac{d}{2}}, \quad \forall n \geq 1$$

such that

$$\rho_n^{\frac{d}{2}} e^{i\theta_n} u(t_n, \rho_n x + x_n) \rightarrow Q \quad \text{strongly in } H_x^{\tilde{s}-},$$

where

$$\tilde{s} := \frac{2d + 8s + s^2 d(2 - \min\{1, \frac{4}{d}\})}{4d + 16s - s^2(d \min\{1, \frac{4}{d}\} + 8)}.$$

In a nutshell, Theorem 1.3 says that up to the symmetries for (1.1), the ground state is the profile for blowup solutions with minimal mass and initial data in H_x^s , $s > s_0(d)$. Alas, we only show this is true along a sequence of times.

To prove Theorems 1.1 through 1.3, we will rely on the *I-method* and Lemma 2.11. The idea behind the *I-method* is to smooth out the initial data in order to access the theory available at H_x^1 regularity. To this end, one introduces the Fourier multiplier I , which is the identity on low frequencies and behaves like a fractional integral operator of order $1 - s$ on high frequencies. Thus, the operator I maps H_x^s to H_x^1 . However, even though we do have energy conservation for (1.1), Iu is not a solution to (1.1) and hence, we expect an energy increment. The key is to prove that on intervals of local well-posedness, the modified energy $E(Iu)$ is an ‘almost conserved’ quantity and grows much slower than the modified kinetic energy $\|\nabla Iu\|_{L_x^2}^2$. This requires delicate estimates on the commutator between I and the nonlinearity. In dimensions one and two, the nonlinearity is algebraic and one can write the commutator explicitly using the Fourier transform and control it by multilinear analysis and bilinear estimates (see [13, 36]). However, in dimensions $d \geq 3$ this method fails. Instead, we will have to rely on more rudimentary tools such as Strichartz and fractional chain rule estimates in order to control the commutator.

The remainder of this paper is organized as follows: In Section 2, we introduce notation and prove some lemmas that will be useful. In Section 3, we revisit the H_x^s local well-posedness theory for (1.1). Section 4 is devoted to controlling the modified energy increment. In Sections 5 through 7 we prove Theorems 1.1 through 1.3.

After this work was submitted, we were informed of an independent paper attacking the same problem, [14]. However, there appear to be several gaps in their argument, for example, in estimating (4.19). While it seems possible to remedy their errors, this would result in a larger value for $s_0(d)$ than that claimed in their paper, which is already inferior to that given here. We contend that all overlap between this paper and [14] can be attributed to the precursors, [13] and [21].

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2. PRELIMINARIES

We will often use the notation $X \lesssim Y$ whenever there exists some constant C so that $X \leq CY$. Similarly, we will use $X \sim Y$ if $X \lesssim Y \lesssim X$. We use $X \ll Y$ if $X \leq cY$ for some small constant c . The derivative operator ∇ refers to the space variable only. We use $A \pm$ to denote $A \pm \varepsilon$ for any sufficiently small $\varepsilon > 0$; the implicit constant in an inequality involving this notation is permitted to depend on ε .

We use $L_x^r(\mathbb{R}^d)$ to denote the Banach space of functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ whose norm

$$\|f\|_r := \left(\int_{\mathbb{R}^d} |f(x)|^r dx \right)^{\frac{1}{r}}$$

is finite, with the usual modifications when $r = \infty$.

We use $L_t^q L_x^r$ to denote the spacetime norm

$$\|u\|_{q,r} := \|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t,x)|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when either q or r are infinity, or when the domain $\mathbb{R} \times \mathbb{R}^d$ is replaced by some smaller spacetime region. When $q = r$ we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

We define the Fourier transform on \mathbb{R}^d to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx.$$

We will make use of the fractional differentiation operators $|\nabla|^s$ defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}_x^s} := \| |\nabla|^s f \|_{L_x^2}$$

and the more common inhomogeneous Sobolev norms

$$\|f\|_{H_x^{s,p}} := \| \langle \nabla \rangle^s f \|_{L_x^p}, \quad \text{where} \quad \langle \nabla \rangle := (1 + |\nabla|^2)^{\frac{1}{2}}.$$

We will often denote $H_x^{s,2}$ by H_x^s .

Let $F(z) := -|z|^{\frac{4}{d}} z$ be the function that defines the nonlinearity in (1.1). Then,

$$F_z(z) := \frac{\partial F}{\partial z}(z) = -\frac{2+d}{d}|z|^{\frac{4}{d}} \quad \text{and} \quad F_{\bar{z}}(z) := \frac{\partial F}{\partial \bar{z}}(z) = -\frac{2}{d}|z|^{\frac{4}{d}} \frac{z}{\bar{z}}.$$

We write F' for the vector $(F_z, F_{\bar{z}})$ and adopt the notation

$$w \cdot F'(z) := w F_z(z) + \bar{w} F_{\bar{z}}(z).$$

In particular, we observe the chain rule

$$\nabla F(u) = \nabla u \cdot F'(u).$$

Clearly $F'(z) = O(|z|^{\frac{4}{d}})$ and we have the Hölder continuity estimate

$$(2.1) \quad |F'(z) - F'(w)| \lesssim |z - w|^{\min\{1, \frac{4}{d}\}} (|z| + |w|)^{\frac{4}{d} - \min\{1, \frac{4}{d}\}}$$

for all $z, w \in \mathbb{C}$. By the Fundamental Theorem of Calculus,

$$F(z + w) - F(z) = \int_0^1 w \cdot F'(z + \theta w) d\theta$$

and hence

$$F(z + w) = F(z) + O(|w||z|^{\frac{4}{d}}) + O(|w|^{\frac{4+d}{d}})$$

for all complex values z and w .

Let $e^{it\Delta}$ be the free Schrödinger propagator. In physical space this is given by the formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) dy$$

for $t \neq 0$ (using a suitable branch cut to define $(4\pi it)^{d/2}$), while in frequency space one can write this as

$$(2.2) \quad \widehat{e^{it\Delta} f}(\xi) = e^{-4\pi^2 it |\xi|^2} \hat{f}(\xi).$$

In particular, the propagator obeys the *dispersive inequality*

$$(2.3) \quad \|e^{it\Delta} f\|_{L_x^\infty} \lesssim |t|^{-\frac{d}{2}} \|f\|_{L_x^1}$$

for all times $t \neq 0$.

We also recall *Duhamel's formula*

$$(2.4) \quad u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} (i u_s + \Delta u)(s) ds.$$

Definition 2.1. A pair of exponents (q, r) is called *Schrödinger-admissible* if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q, r \leq \infty, \quad \text{and} \quad (q, r, d) \neq (2, \infty, 2).$$

Throughout this paper we will use the following admissible pairs:

$$(2, \frac{2d}{d-2}) \quad \text{and} \quad (\gamma, \rho) := (\frac{2(d+2)}{d-2s}, \frac{2d(d+2)}{d^2+4s}) \quad \text{with } 0 < s < 1.$$

Let $\rho^* := \frac{2(d+2)}{d-2s}$. Using Hölder and Sobolev embedding, on any spacetime slab $I \times \mathbb{R}^d$ we estimate

$$(2.5) \quad \|F(u)\|_{\gamma', \rho'} \lesssim |I|^{\frac{2s}{d}} \|u\|_{\gamma, \rho} \|u\|_{\gamma, \rho^*}^{\frac{4}{d}} \lesssim |I|^{\frac{2s}{d}} \|\langle \nabla \rangle^s u\|_{\gamma, \rho}^{1+\frac{4}{d}}.$$

Let $I \times \mathbb{R}^d$ be a spacetime slab; we define the Strichartz norm

$$\|u\|_{S^0(I)} := \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)}.$$

We define the Strichartz space $S^0(I)$ to be the closure of all test functions under the Strichartz norm $\|\cdot\|_{S^0(I)}$. We use $N^0(I)$ to denote the dual space of $S^0(I)$.

We record the standard Strichartz estimates which we will invoke repeatedly throughout this paper (see [33, 17] for (q, r) admissible with $q > 2$ and [24] for the endpoint $(2, \frac{2d}{d-2})$):

Lemma 2.2. *Let I be a compact time interval, $t_0 \in I$, $s \geq 0$, and let u be a solution to the forced Schrödinger equation*

$$iu_t + \Delta u = \sum_{i=1}^m F_i$$

for some functions F_1, \dots, F_m . Then,

$$(2.6) \quad \|\nabla|^s u\|_{S^0(I)} \lesssim \|u(t_0)\|_{\dot{H}^s(\mathbb{R}^d)} + \sum_{i=1}^m \|\nabla|^s F_i\|_{L_t^{q'_i} L_x^{r'_i}(I \times \mathbb{R}^d)}$$

for any admissible pairs (q_i, r_i) , $1 \leq i \leq m$.

We will also need some Littlewood-Paley theory. Specifically, let $\varphi(\xi)$ be a smooth bump supported in the ball $|\xi| \leq 2$ and equalling one on the ball $|\xi| \leq 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$ we define the Littlewood-Paley operators

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N) \hat{f}(\xi), \\ \widehat{P_{> N} f}(\xi) &:= [1 - \varphi(\xi/N)] \hat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= [\varphi(\xi/N) - \varphi(2\xi/N)] \hat{f}(\xi). \end{aligned}$$

Similarly we can define $P_{< N}$, $P_{\geq N}$, and $P_{M < \cdot \leq N} := P_{\leq N} - P_{\leq M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\leq N}$ for $P_{\leq N} f$ and similarly for the other operators. We recall the following standard Bernstein and Sobolev type inequalities:

Lemma 2.3. *For any $1 \leq p \leq q \leq \infty$ and $s > 0$, we have*

$$\begin{aligned} \|P_{\geq N} f\|_{L_x^p} &\lesssim N^{-s} \|\nabla|^s P_{\geq N} f\|_{L_x^p} \\ \|\nabla|^s P_{\leq N} f\|_{L_x^p} &\lesssim N^s \|P_{\leq N} f\|_{L_x^p} \\ \|\nabla|^{\pm s} P_N f\|_{L_x^p} &\sim N^{\pm s} \|P_N f\|_{L_x^p} \\ \|P_{\leq N} f\|_{L_x^q} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq N} f\|_{L_x^p} \\ \|P_N f\|_{L_x^q} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_N f\|_{L_x^p}. \end{aligned}$$

For $N > 1$, we define the Fourier multiplier I_N (cf. [10]) by

$$\widehat{I_N u}(\xi) := m_N(\xi) \hat{u}(\xi),$$

where m_N is a smooth radial decreasing function such that

$$m_N(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq N \\ (\frac{|\xi|}{N})^{s-1}, & \text{if } |\xi| \geq 2N. \end{cases}$$

Thus, I_N is the identity operator on frequencies $|\xi| \leq N$ and behaves like a fractional integral operator of order $1-s$ on higher frequencies. In particular, I_N maps H_x^s to H_x^1 ; this allows us to access the theory available for H_x^1 data. We collect the basic properties of I_N into the following:

Lemma 2.4. *Let $1 < p < \infty$ and $0 \leq \sigma \leq s < 1$. Then,*

$$(2.7) \quad \|I_N f\|_p \lesssim \|f\|_p$$

$$(2.8) \quad \|\nabla|^\sigma P_{> N} f\|_p \lesssim N^{\sigma-1} \|\nabla I_N f\|_p$$

$$(2.9) \quad \|f\|_{H_x^s} \lesssim \|I_N f\|_{H_x^1} \lesssim N^{1-s} \|f\|_{H_x^s}.$$

Proof. The estimate (2.7) is a direct consequence of the multiplier theorem.

To prove (2.8), we write

$$\| |\nabla|^\sigma P_{>N} f \|_p = \| P_{>N} |\nabla|^\sigma (\nabla I_N)^{-1} \nabla I_N f \|_p.$$

The claim follows again from the multiplier theorem.

Now we turn to (2.9). By the definition of the operator I_N and (2.8),

$$\begin{aligned} \| f \|_{H_x^s} &\lesssim \| P_{\leq N} f \|_{H_x^s} + \| P_{>N} f \|_2 + \| |\nabla|^s P_{>N} f \|_2 \\ &\lesssim \| P_{\leq N} I_N f \|_{H_x^1} + N^{-1} \| \nabla I_N f \|_2 + N^{s-1} \| \nabla I_N f \|_2 \\ &\lesssim \| I_N f \|_{H_x^1}. \end{aligned}$$

On the other hand, since the operator I_N commutes with $\langle \nabla \rangle^s$, we get

$$\| I_N f \|_{H_x^1} = \| \langle \nabla \rangle^{1-s} I_N \langle \nabla \rangle^s f \|_2 \lesssim N^{1-s} \| \langle \nabla \rangle^s f \|_2 \lesssim N^{1-s} \| f \|_{H_x^s},$$

which proves the last inequality in (2.9). Note that a similar argument also yields

$$(2.10) \quad \| I_N f \|_{\dot{H}_x^1} \lesssim N^{1-s} \| f \|_{\dot{H}_x^s}.$$

□

The estimate (2.8) shows that we can control the high frequencies of a function f in the Sobolev space $H_x^{\sigma,p}$ by the smoother function $I_N f$ in a space with a loss of derivative but a gain of negative power of N . This fact is crucial in extracting the negative power of N when estimating the increment of the modified Hamiltonian.

In dimensions one and two, one can use multilinear analysis to understand commutator expressions like $F(I_N u) - I_N F(u)$; on the Fourier side, one can expand this commutator into a product of Fourier transforms of u and $I_N u$ and carefully measure the frequency interactions to derive an estimate (see for example [13, 36]). However, this is not possible in dimensions $d \geq 3$. Instead, we will have to rely on the following rougher (weaker, but more robust) lemma:

Lemma 2.5. *Let $1 < r, r_1, r_2 < \infty$ be such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and let $0 < \nu < s$. Then,*

$$(2.11) \quad \| I_N(fg) - (I_N f)g \|_r \lesssim N^{-(1-s+\nu)} \| I_N f \|_{r_1} \| \langle \nabla \rangle^{1-s+\nu} g \|_{r_2}.$$

Proof. Applying a Littlewood-Paley decomposition to f and g , we write

$$\begin{aligned} I_N(fg) - (I_N f)g &= I_N(fg_{\leq 1}) - (I_N f)g_{\leq 1} + \sum_{1 < M \in 2^{\mathbb{Z}}} [I_N(f_{\lesssim M} g_M) - (I_N f_{\lesssim M})g_M] \\ &\quad + \sum_{1 < M \in 2^{\mathbb{Z}}} [I_N(f_{\gg M} g_M) - (I_N f_{\gg M})g_M] \\ &= I_N(f_{\gtrsim N} g_{\leq 1}) - (I_N f_{\gtrsim N})g_{\leq 1} + \sum_{N \lesssim M \in 2^{\mathbb{Z}}} [I_N(f_{\lesssim M} g_M) - (I_N f_{\lesssim M})g_M] \\ &\quad + \sum_{1 < M \in 2^{\mathbb{Z}}} [I_N(f_{\gg M} g_M) - (I_N f_{\gg M})g_M] \\ (2.12) \quad &= I + II + III. \end{aligned}$$

The second equality above follows from the fact that the operator I_N is the identity operator on frequencies $|\xi| \leq N$; thus,

$$I_N(f_{\ll N} g_{\leq 1}) = (I_N f_{\ll N})g_{\leq 1} \quad \text{and} \quad I_N(f_{\lesssim M} g_M) = (I_N f_{\lesssim M})g_M \quad \text{for all } M \ll N.$$

We first consider *II*. Dropping the operator I_N , by Hölder and Bernstein we estimate

$$\begin{aligned} \|I_N(f_{\lesssim M} g_M) - (I_N f_{\lesssim M}) g_M\|_r &\lesssim \|f_{\lesssim M}\|_{r_1} \|g_M\|_{r_2} \\ &\lesssim \left(\frac{M}{N}\right)^{1-s} \|I_N f\|_{r_1} \|g_M\|_{r_2} \\ &\lesssim M^{-\nu} N^{-(1-s)} \|I_N f\|_{r_1} \|\nabla|^{1-s+\nu} g\|_{r_2}. \end{aligned}$$

Summing over all M such that $N \lesssim M \in 2^{\mathbb{Z}}$, we get

$$(2.13) \quad II \lesssim N^{-(1-s+\nu)} \|I_N f\|_{r_1} \|\nabla|^{1-s+\nu} g\|_{r_2}.$$

We turn now towards *III*. Applying a Littlewood-Paley decomposition to f , we write each term in *III* as

$$\begin{aligned} I_N(f_{\gg M} g_M) - (I_N f_{\gg M}) g_M &= \sum_{1 \ll k \in \mathbb{N}} [I_N(f_{2^k M} g_M) - (I_N f_{2^k M}) g_M] \\ &= \sum_{\substack{1 \ll k \in \mathbb{N} \\ N \lesssim 2^k M}} [I_N(f_{2^k M} g_M) - (I_N f_{2^k M}) g_M]. \end{aligned}$$

To derive the second inequality, we used again the fact that the operator I_N is the identity on frequencies $|\xi| \leq N$.

We write

$$[I_N(f_{2^k M} g_M) - (I_N f_{2^k M}) g_M]^\sim(\xi) = \int_{\xi=\xi_1+\xi_2} (m_N(\xi_1+\xi_2) - m_N(\xi_1)) \widehat{f_{2^k M}}(\xi_1) \widehat{g_M}(\xi_2).$$

For $|\xi_1| \sim 2^k M$, $k \gg 1$, and $|\xi_2| \sim M$, the Fundamental Theorem of Calculus implies

$$|m_N(\xi_1 + \xi_2) - m_N(\xi_1)| \lesssim 2^{-k} \left(\frac{2^k M}{N}\right)^{s-1}.$$

By the Coifman-Meyer multilinear multiplier theorem, [8, 9], and Bernstein, we get

$$\begin{aligned} \|I_N(f_{2^k M} g_M) - (I_N f_{2^k M}) g_M\|_r &\lesssim 2^{-k} \left(\frac{2^k M}{N}\right)^{s-1} \|f_{2^k M}\|_{r_1} \|g_M\|_{r_2} \\ &\lesssim 2^{-k} M^{-(1-s+\nu)} \|I_N f\|_{r_1} \|\nabla|^{1-s+\nu} g\|_{r_2}. \end{aligned}$$

Summing over M and k such that $N \lesssim 2^k M$, and recalling that $0 < \nu < s$, we get

$$(2.14) \quad III \lesssim N^{-(1-s+\nu)} \|I_N f\|_{r_1} \|\nabla|^{1-s+\nu} g\|_{r_2}.$$

To estimate I , we apply the same argument as for *III*. We get

$$\begin{aligned} I = \|I_N(f_{\gtrsim N} g_{\leq 1}) - (I_N f_{\gtrsim N}) g_{\leq 1}\|_r &\lesssim \sum_{k \in \mathbb{N}, 2^k \gtrsim N} \|I_N(f_{2^k} g_{\leq 1}) - (I_N f_{2^k}) g_{\leq 1}\|_r \\ &\lesssim \sum_{k \in \mathbb{N}, 2^k \gtrsim N} 2^{-k} \|I_N f\|_{r_1} \|g\|_{r_2} \\ (2.15) \quad &\lesssim N^{-1} \|I_N f\|_{r_1} \|g\|_{r_2}. \end{aligned}$$

Putting (2.12) through (2.15) together, we derive (2.11). \square

As an application of Lemma 2.5 we have the following commutator estimate:

Lemma 2.6. *Let $1 < r, r_1, r_2 < \infty$ be such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then, for any $0 < \nu < s$ we have*

$$(2.16) \quad \|\nabla I_N F(u) - (I_N \nabla u) F'(u)\|_r \lesssim N^{-1+s-\nu} \|\nabla I_N u\|_{r_1} \|\langle \nabla \rangle^{1-s+\nu} F'(u)\|_{r_2}$$

$$(2.17) \quad \|\nabla I_N F(u)\|_r \lesssim \|\nabla I_N u\|_{r_1} \|F'(u)\|_{r_2} + N^{-1+s-\nu} \|\nabla I_N u\|_{r_1} \|\langle \nabla \rangle^{1-s+\nu} F'(u)\|_{r_2}.$$

Proof. As

$$\nabla F(u) = \nabla u \cdot F'(u),$$

the estimate (2.16) follows immediately from Lemma 2.5 with $f := \nabla u$ and $g := F'(u)$. The estimate (2.17) is a consequence of (2.16) and the triangle inequality. \square

Since we work at regularity $0 < s < 1$, we will need the following fractional chain rules to estimate our nonlinearity in H_x^s .

Lemma 2.7 (Fractional chain rule for a C^1 function). *Suppose that $F \in C^1(\mathbb{C})$, $\sigma \in (0, 1)$, and $1 < r, r_1, r_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then,*

$$\| |\nabla|^\sigma F(u) \|_r \lesssim \|F'(u)\|_{r_1} \| |\nabla|^\sigma u \|_{r_2}.$$

Lemma 2.8 (Fractional chain rule for a Lipschitz function). *Let F be a Lipschitz function, $\sigma \in (0, 1)$, and $1 < r < \infty$. Then,*

$$\| |\nabla|^\sigma F(u) \|_r \lesssim \|F'\|_\infty \| |\nabla|^\sigma u \|_r.$$

Lemma 2.9 (Fractional derivatives for fractional powers). *Let F be a Hölder continuous function of order $0 < \alpha < 1$. Then, for every $0 < \sigma < \alpha$, $1 < r < \infty$, and $\frac{\sigma}{\alpha} < \delta < 1$ we have*

$$(2.18) \quad \| |\nabla|^\sigma F(u) \|_r \lesssim \| |u|^{\alpha-\frac{\sigma}{\alpha}} \|_{r_1} \| |\nabla|^\delta u \|_{\frac{\sigma}{\delta} r_2},$$

provided $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and $(1 - \frac{\sigma}{\alpha\delta})r_1 > 1$.

The first two results originate in [7] and [32]; for a textbook treatment see [35]. The third result can be found in Appendix A of [38]. Using the chain rule estimates in these lemmas, we can upgrade the pointwise in time commutator estimate in Lemma 2.6 to a spacetime estimate:

Lemma 2.10. *Let I be a compact time interval and let $\frac{1}{1+\min\{1, \frac{4}{d}\}} < s < 1$. Then,*

$$(2.19) \quad \|(I_N \nabla u) F'(u) - \nabla I_N F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I \times \mathbb{R}^d)} \lesssim N^{-\min\{1, \frac{4}{d}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0(I)}^{1+\frac{4}{d}}$$

$$(2.20) \quad \|\langle \nabla \rangle I_N F(u)\|_{N^0(I)} \lesssim |I|^{\frac{2s}{d}} \|\langle \nabla \rangle I_N u\|_{S^0(I)}^{1+\frac{4}{d}} + N^{-\min\{1, \frac{4}{d}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0(I)}^{1+\frac{4}{d}}.$$

Proof. Throughout the proof, all spacetime norms will be taken on the slab $I \times \mathbb{R}^d$.

As by hypothesis $s > \frac{1}{1+\min\{1, \frac{4}{d}\}}$, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have $s > \frac{1+\varepsilon}{1+\min\{1, \frac{4}{d}\}}$. Let $\nu := \min\{1, \frac{4}{d}\}s - (1-s) - \varepsilon$ with $0 < \varepsilon < \varepsilon_0$. It is easy to check that $0 < \nu < s$. Applying Lemma 2.6 with this value of ν , we get

$$\begin{aligned} & \|(\nabla I_N u) F'(u) - \nabla I_N F(u)\|_{2, \frac{2d}{d+2}} \\ & \lesssim N^{-\min\{1, \frac{4}{d}\}s+\varepsilon} \|\nabla I_N u\|_{2, \frac{2d}{d+2}} \|\langle \nabla \rangle^{\min\{1, \frac{4}{d}\}s-\varepsilon} F'(u)\|_{\infty, \frac{d}{2}}. \end{aligned}$$

The claim (2.19) will follow immediately from the estimate above, provided we show

$$(2.21) \quad \|\langle \nabla \rangle^{\min\{1, \frac{4}{d}\}s-\varepsilon} F'(u)\|_{\infty, \frac{d}{2}} \lesssim \|\langle \nabla \rangle I_N u\|_{\infty, 2}^{\frac{4}{d}}.$$

We start by observing that for any (q, r) admissible pair,

$$(2.22) \quad \|\langle \nabla \rangle^s u\|_{q, r} \lesssim \|\langle \nabla \rangle I_N u\|_{q, r}.$$

Indeed, decomposing $u := u_{\leq N} + u_{> N}$ and using Lemma 2.4 and the fact that I_N is the identity operator on frequencies $|\xi| \leq N$, we get

$$\begin{aligned} \|u\|_{q, r} &\leq \|u_{\leq N}\|_{q, r} + \|u_{> N}\|_{q, r} \\ &\lesssim \|I_N u_{\leq N}\|_{q, r} + N^{-1} \|\nabla I_N u_{> N}\|_{q, r} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{q, r}. \end{aligned}$$

Similarly, we estimate

$$\begin{aligned} \|\nabla^s u\|_{q, r} &\leq \|\nabla^s u_{\leq N}\|_{q, r} + \|\nabla^s u_{> N}\|_{q, r} \\ &\lesssim \|\nabla^s I_N u_{\leq N}\|_{q, r} + N^{s-1} \|\nabla I_N u_{> N}\|_{q, r} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{q, r}, \end{aligned}$$

and the estimate (2.22) follows.

As $F'(u) = O(|u|^{\frac{4}{d}})$, by (2.22) we get

$$(2.23) \quad \|F'(u)\|_{\infty, \frac{d}{2}} \lesssim \|\langle \nabla \rangle I_N u\|_{\infty, 2}^{\frac{4}{d}}.$$

We first prove (2.21) for $d \leq 4$. By (2.23), we estimate

$$\begin{aligned} \|\langle \nabla \rangle^{\min\{1, \frac{4}{d}\}s-\varepsilon} F'(u)\|_{\infty, \frac{d}{2}} &= \|\langle \nabla \rangle^{s-\varepsilon} F'(u)\|_{\infty, \frac{d}{2}} \\ &\lesssim \|\langle \nabla \rangle^s F'(u)\|_{\infty, \frac{d}{2}} \\ &\lesssim \|F'(u)\|_{\infty, \frac{d}{2}} + \|\nabla^s F'(u)\|_{\infty, \frac{d}{2}} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{\infty, 2}^{\frac{4}{d}} + \|\nabla^s F'(u)\|_{\infty, \frac{d}{2}}. \end{aligned}$$

Using Lemmas 2.7 (for $d = 3$) and 2.8 (for $d = 4$) together with (2.22), we estimate

$$\|\nabla^s F'(u)\|_{\infty, \frac{d}{2}} \lesssim \|\nabla^s u\|_{\infty, 2} \|u\|_{\infty, 2}^{\frac{4}{d}-1} \lesssim \|\langle \nabla \rangle I_N u\|_{\infty, 2}^{\frac{4}{d}}.$$

Thus, (2.21) holds for $d \leq 4$.

If $d > 4$, using Lemma 2.9 (with $\alpha := \frac{4}{d}$, $\sigma := \frac{4s}{d} - \varepsilon$, and $\delta := s$) and (2.22), we get

$$\begin{aligned} \|\nabla^{\min\{1, \frac{4}{d}\}s-\varepsilon} F'(u)\|_{\infty, \frac{d}{2}} &= \|\nabla^{\frac{4s}{d}-\varepsilon} F'(u)\|_{\infty, \frac{d}{2}} \lesssim \|u\|_{\infty, 2}^{\frac{\varepsilon}{s}} \|\nabla^s u\|_{\infty, 2}^{\frac{4}{d}-\frac{\varepsilon}{s}} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{\infty, 2}^{\frac{4}{d}}. \end{aligned}$$

From this and (2.23) we derive (2.21) in the case $d > 4$. The claim (2.19) follows.

We now consider (2.20). Using (2.5), (2.7), and (2.22), we obtain

$$\|I_N F(u)\|_{N^0(I)} \lesssim \|F(u)\|_{\gamma', \rho'} \lesssim |I|^{\frac{2s}{d}} \|\langle \nabla \rangle^s u\|_{\gamma, \rho}^{1+\frac{4}{d}} \lesssim |I|^{\frac{2s}{d}} \|\langle \nabla \rangle I_N u\|_{\gamma, \rho}^{1+\frac{4}{d}}.$$

Similarly, by Hölder and (2.22),

$$\begin{aligned}
\|(I_N \nabla u) F'(u)\|_{N^0(I)} &\lesssim \|(I_N \nabla u) F'(u)\|_{\gamma', \rho'} \\
&\lesssim |I|^{\frac{2s}{d}} \|I_N \nabla u\|_{\gamma, \rho} \|u\|_{\gamma, \rho^*}^{\frac{4}{d}} \\
&\lesssim |I|^{\frac{2s}{d}} \|I_N \nabla u\|_{\gamma, \rho} \|\langle \nabla \rangle^s u\|_{\gamma, \rho}^{\frac{4}{d}} \\
&\lesssim |I|^{\frac{2s}{d}} \|\langle \nabla \rangle I_N u\|_{\gamma, \rho}^{1+\frac{4}{d}}.
\end{aligned}$$

The estimate (2.20) follows from the estimates above, (2.19) and the triangle inequality. \square

We end this section with the following concentration-compactness lemma:

Lemma 2.11 (Concentration-compactness, [20]). *Let $\{v_n\}_{n \geq 1}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ such that*

$$\limsup_{n \rightarrow \infty} \|\nabla v_n\|_2 \leq M < \infty$$

and

$$\limsup_{n \rightarrow \infty} \|v_n\|_{2+\frac{4}{d}} \geq m > 0.$$

Then, there exists $\{x_n\}_{n \geq 1} \subset \mathbb{R}^d$ such that, up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly in } H_x^1 \text{ as } n \rightarrow \infty$$

with $\|V\|_2 \geq \sqrt{\frac{d}{d+2} \frac{m^{\frac{d}{2}+1}}{M}} \|Q\|_2$.

3. LOCAL H_x^s THEORY

In this section, we review the local H_x^s theory for the equation (1.1) as described in [6].

Proposition 3.1 (Local well-posedness in H_x^s , [6]). *Let $0 < s < 1$ and $u_0 \in H^s(\mathbb{R}^d)$. Then, the equation (1.1) is wellposed on $[0, T_{lp}]$ with*

$$T_{lp} = C_0 \|\langle \nabla \rangle^s u_0\|_2^{-\frac{2}{s}}.$$

Moreover, the unique solution u enjoys the following estimate:

$$\|\langle \nabla \rangle^s u\|_{S^0([0, T_{lp}])} \lesssim \|\langle \nabla \rangle^s u_0\|_2.$$

Here, C_0 and the implicit constant depend only on the dimension d and the regularity s .

As a direct consequence of the H_x^s local well-posedness result, we have the following lower bound on the blowup rate of the H_x^s -norm:

Corollary 3.2 (Blowup criterion, [6]). *Let $0 < s < 1$ and $u_0 \in H_x^s$. Assume that the unique solution u to (1.1) blows up at time $0 < T^* < \infty$. Then, there exists a constant C depending only on d and s such that*

$$\|u(t)\|_{H_x^s} \geq C(T^* - t)^{-\frac{s}{2}}.$$

As a variation of Proposition 3.1, we have

Proposition 3.3 ('Modified' local well-posedness). *Let $\frac{\frac{4}{d}+1}{1+\min\{1, \frac{4}{d}\}+\frac{4}{d}} < s < 1$ and $u_0 \in H_x^s$. Let also*

$$(3.1) \quad N \gg \|u_0\|_{H_x^s}^{\frac{4}{\min\{4,d\}s-(4+d)(1-s)-d\varepsilon}} \quad \text{for any } \varepsilon > 0 \text{ sufficiently small}$$

$$(3.2) \quad \tilde{T}_{lwp} := c_0 \|\langle \nabla \rangle I_N u_0\|_2^{-\frac{2}{s}} \quad \text{for a small constant } c_0 = c_0(d, s).$$

Then, (1.1) is wellposed on $[0, \tilde{T}_{lwp}]$ and moreover

$$(3.3) \quad \|\langle \nabla \rangle I_N u\|_{S^0([0, \tilde{T}_{lwp}])} \lesssim \|\langle \nabla \rangle I_N u_0\|_2.$$

Proof. As by Lemma 2.4,

$$\|u_0\|_{H_x^s} \lesssim \|\langle \nabla \rangle I_N u_0\|_2,$$

choosing c_0 sufficiently small, we get

$$\tilde{T}_{lwp} \leq T_{lwp}.$$

Thus, (1.1) is wellposed in H_x^s on $[0, \tilde{T}_{lwp}]$. Let u be the unique solution; by Proposition 3.1 we have

$$(3.4) \quad \|\langle \nabla \rangle^s u\|_{S^0([0, \tilde{T}_{lwp}])} \lesssim \|u_0\|_{H_x^s}.$$

On the other hand, by Strichartz, for any $t \leq \tilde{T}_{lwp}$ we estimate

$$\|\langle \nabla \rangle I_N u\|_{S^0([0, t])} \lesssim \|\langle \nabla \rangle I_N u_0\|_2 + \|\langle \nabla \rangle I_N F(u)\|_{N^0([0, t])}.$$

As by hypothesis, $s > \frac{\frac{4}{d}+1}{1+\min\{1, \frac{4}{d}\}+\frac{4}{d}} > \frac{1}{1+\min\{1, \frac{4}{d}\}}$, by Lemma 2.10 we get

$$(3.5) \quad \begin{aligned} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])} &\lesssim \|\langle \nabla \rangle I_N u_0\|_2 + t^{\frac{2s}{d}} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1+\frac{4}{d}} \\ &\quad + N^{-\min\{1, \frac{4}{d}\}s+\varepsilon} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1+\frac{4}{d}} \end{aligned}$$

for any $\varepsilon > 0$ sufficiently small. For N satisfying (3.1), we use Lemma 2.4 and (3.4) to estimate

$$\begin{aligned} &N^{-\min\{1, \frac{4}{d}\}s+\varepsilon} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1+\frac{4}{d}} \\ &\lesssim N^{-\min\{1, \frac{4}{d}\}s+\varepsilon} N^{(1+\frac{4}{d})(1-s)} \|\langle \nabla \rangle^s u\|_{S^0([0, t])}^{1+\frac{4}{d}} \\ &\ll \|u_0\|_{H_x^s}^{\frac{4}{\min\{4,d\}s-(4+d)(1-s)-d\varepsilon} \times [-\min\{1, \frac{4}{d}\}s+\varepsilon+(1+\frac{4}{d})(1-s)]} \|u_0\|_{H_x^s}^{1+\frac{4}{d}} \\ &\lesssim \|u_0\|_{H_x^s}^{-\frac{4}{d}} \|u_0\|_{H_x^s}^{1+\frac{4}{d}} \lesssim \|\langle \nabla \rangle I_N u_0\|_2. \end{aligned}$$

Here, we have used the fact that $s > \frac{\frac{4}{d}+1}{1+\min\{1, \frac{4}{d}\}+\frac{4}{d}}$ implies that there exists $\varepsilon_1 > 0$ sufficiently small such that for any $0 < \varepsilon < \varepsilon_1$ we have $s > \frac{\frac{4}{d}+1+\varepsilon}{1+\min\{1, \frac{4}{d}\}+\frac{4}{d}}$; thus the power of N in the estimates above is negative for $0 < \varepsilon < \varepsilon_1$.

Returning to (3.5), we conclude that

$$\|\langle \nabla \rangle I_N u\|_{S^0([0, t])} \lesssim \|\langle \nabla \rangle I_N u_0\|_2 + t^{\frac{2s}{d}} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1+\frac{4}{d}}.$$

Standard arguments yield (3.3), provided

$$t \leq \tilde{T}_{lwp} = c_0(d, s) \|\langle \nabla \rangle I_N u_0\|_2^{-\frac{2}{s}}.$$

□

4. MODIFIED ENERGY INCREMENT

The main purpose of this section is to prove that the modified energy of u , $E(I_N u)$, grows much slower than the modified kinetic energy of u , $\|\nabla I_N u\|_2^2$. As will be shown later, this result is crucial in establishing the main theorems.

Before stating the result, we need to introduce more notation. We define

$$\Lambda(t) := \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{H_x^s} \quad \text{and} \quad \Sigma(t) := \sup_{0 \leq \tau \leq t} \|I_N u(\tau)\|_{H_x^1}.$$

With this notation we have the following

Proposition 4.1 (Increment of the modified energy). *Let $s_0(d) < s < 1$ and let $u_0 \in H_x^s$ such that the corresponding solution u to (1.1) blows up at time $0 < T^* < \infty$. Let $0 < T < T^*$. Then, for*

$$(4.1) \quad N(T) := C\Lambda(T)^{\frac{p(s)}{2(1-s)}},$$

we have

$$|E(I_{N(T)} u(T))| \lesssim \Lambda(T)^{p(s)}.$$

Here, C and the implicit constant depend only on s , T^* , and $\|u_0\|_{H_x^s}$, and $p(s)$ is given by

$$p(s) := \frac{2(2 + \frac{2}{s} + \frac{8}{d})(1-s)}{\min\{1, \frac{4}{d}\}s - (\frac{2}{s} + \frac{8}{d})(1-s)}.$$

Note that by Lemma 2.4, $\Lambda(T) \lesssim \Sigma(T)$. Thus, if the solution blows up at time T^* , the modified energy $E(I_{N(T)} u(T))$ is at most $O(\Sigma(T)^{p(s)})$, which is much smaller than the modified kinetic energy, $\|\nabla I_{N(T)} u(T)\|_2^2 = O(\Sigma(T)^2)$ for $s > s_0(d)$.

We prove Proposition 4.1 in two steps. The first step is to control the increment of the modified energy of u on intervals of local well-posedness $[0, \tilde{T}_{lwp}]$. The second step is to divide the interval $[0, T]$ into finitely many subintervals of local well-posedness, control the increment of the modified energy of u on each of these subintervals, and sum these bounds.

We start with the following

Lemma 4.2 (Local increment of the modified energy). *Let $\frac{1+\frac{4}{d}}{1+\frac{4}{d}+\min\{1, \frac{4}{d}\}} < s < 1$ and let $u_0 \in H_x^s$. Assume that N and \tilde{T}_{lwp} satisfy (3.1) and (3.2) respectively, that is,*

$$N \gg \|u_0\|_{H_x^s}^{\frac{4}{\min\{4, d\}s - (4+d)(1-s)}} \\ \tilde{T}_{lwp} := c_0 \|\langle \nabla \rangle I_N u_0\|_2^{-\frac{2}{s}} \quad \text{for a small constant } c_0 = c_0(d, s).$$

Then,

$$\sup_{t \in [0, \tilde{T}_{lwp}]} |E(I_N u(t))| \leq |E(I_N u_0)| + CN^{-\min\{1, \frac{4}{d}\}s + (\|I_N \langle \nabla \rangle u_0\|_2^{2+\frac{4}{d}} + \|I_N \langle \nabla \rangle u_0\|_2^{2+\frac{8}{d}})}.$$

Here, the constant C depends on s , T^* , and $\|u_0\|_{H_x^s}$.

Proof. Note that by Proposition 3.3, (1.1) is wellposed on $[0, \tilde{T}_{lwp}]$. Furthermore, the unique solution u to (1.1) on $[0, \tilde{T}_{lwp}]$ satisfies

$$(4.2) \quad \|\langle \nabla \rangle I_N u\|_{S^0([0, \tilde{T}_{lwp}])} \lesssim \|\langle \nabla \rangle I_N u_0\|_2.$$

Let $0 < t \leq \tilde{T}_{lwp}$; throughout the rest of the proof, all spacetime norms will be taken on $[0, t] \times \mathbb{R}^d$. By the Fundamental Theorem of Calculus, we can write the modified energy increment as

$$\begin{aligned} E(I_N u(t)) - E(I_N u_0) &= \int_0^t \frac{\partial}{\partial s} E(I_N u(s)) ds \\ &= \operatorname{Re} \int_0^t \int_{\mathbb{R}^d} \overline{I_N u_t} (-\Delta I_N u + F(I_N u)) dx ds. \end{aligned}$$

As $I_N u_t = i\Delta I_N u - iI_N F(u)$, we have

$$\operatorname{Re} \int_0^t \int_{\mathbb{R}^d} \overline{I_N u_t} (-\Delta I_N u + I_N F(u)) dx ds = 0.$$

Thus, after an integration by parts,

$$\begin{aligned} E(I_N u(t)) - E(I_N u_0) &= \operatorname{Re} \int_0^t \int_{\mathbb{R}^d} \overline{I_N u_t} [F(I_N u) - I_N F(u)] dx ds \\ (4.3) \quad &= -\operatorname{Im} \int_0^t \int_{\mathbb{R}^d} \overline{\nabla I_N u} \cdot \nabla [F(I_N u) - I_N F(u)] dx ds \end{aligned}$$

$$(4.4) \quad -\operatorname{Im} \int_0^t \int_{\mathbb{R}^d} \overline{I_N F(u)} \cdot [F(I_N u) - I_N F(u)] dx ds.$$

Consider the contribution from (4.3). By the triangle inequality,

$$\begin{aligned} \|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}} &\lesssim \|(\nabla I_N u)[F'(I_N u) - F'(u)]\|_{2, \frac{2d}{d+2}} \\ &\quad + \|(\nabla I_N u)F'(u) - \nabla I_N F(u)\|_{2, \frac{2d}{d+2}}. \end{aligned}$$

By Hölder, (2.1), (2.8), and (2.22), we estimate

$$\begin{aligned} &\|(\nabla I_N u)[F'(I_N u) - F'(u)]\|_{2, \frac{2d}{d+2}} \\ &\lesssim \|\nabla I_N u\|_{2, \frac{2d}{d-2}} \|F'(I_N u) - F'(u)\|_{\infty, \frac{d}{2}} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{S^0([0, t])} \| |I_N u - u|^{\min\{1, \frac{4}{d}\}} (|I_N u| + |u|)^{\frac{4}{d} - \min\{1, \frac{4}{d}\}} \|_{\infty, \frac{d}{2}} \\ &\lesssim \|\langle \nabla \rangle I_N u\|_{S^0([0, t])} \|P_{>N} u\|_{\infty, 2}^{\min\{1, \frac{4}{d}\}} \|u\|_{\infty, 2}^{\frac{4}{d} - \min\{1, \frac{4}{d}\}} \\ &\lesssim N^{-\min\{1, \frac{4}{d}\}} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1+\frac{4}{d}}. \end{aligned}$$

Combining this with (2.19), we get

$$(4.5) \quad \|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}} \lesssim N^{-\min\{1, \frac{4}{d}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1+\frac{4}{d}}.$$

Therefore

$$\begin{aligned} |4.3| &\lesssim \|\nabla I_N u\|_{2, \frac{2d}{d-2}} \|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}} \\ (4.6) \quad &\lesssim N^{-\min\{1, \frac{4}{d}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{2+\frac{4}{d}}. \end{aligned}$$

We turn now toward (4.4). By (4.5) and Sobolev embedding, we estimate

$$\begin{aligned} |4.4| &\lesssim \|\nabla [F(I_N u) - I_N F(u)]\|_{2, \frac{2d}{d+2}} \| |\nabla|^{-1} I_N F(u) \|_{2, \frac{2d}{d-2}} \\ (4.7) \quad &\lesssim N^{-\min\{1, \frac{4}{d}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0([0, t])}^{1+\frac{4}{d}} \|I_N F(u)\|_{2, 2}. \end{aligned}$$

To estimate the last factor in (4.7), we drop the operator I_N and use Sobolev embedding to obtain

$$\|I_N F(u)\|_{2,2} \lesssim \|u\|_{\frac{2(d+4)}{d}, \frac{2(d+4)}{d+2}}^{1+\frac{4}{d}} \lesssim \|\nabla|^{\frac{d}{d+4}} u\|_{\frac{2(d+4)}{d}, \frac{2(d+4)}{d+2}}^{1+\frac{4}{d}}.$$

Note that $(\frac{2(d+4)}{d}, \frac{2(d+4)}{d+2})$ is a Schrödinger admissible pair. Decompose $u := u_{\leq N} + u_{>N}$. To estimate the low frequencies, we use the fact that I_N is the identity on frequencies $|\xi| \leq N$:

$$\|\nabla|^{\frac{d}{d+4}} u_{\leq N}\|_{\frac{2(d+4)}{d}, \frac{2(d+4)}{d+2}} \lesssim \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}.$$

For the high frequencies, we use Lemma 2.4 to get

$$\|\nabla|^{\frac{d}{d+4}} u_{>N}\|_{\frac{2(d+4)}{d}, \frac{2(d+4)}{d+2}} \lesssim N^{-\frac{4}{d+4}} \|\nabla I_N u_{>N}\|_{S^0([0,t])},$$

provided $s > \frac{d}{d+4}$; this condition is satisfied since by assumption,

$$s > \frac{1+\frac{4}{d}}{1+\frac{4}{d}+\min\{1, \frac{4}{d}\}} > \frac{d}{d+4}.$$

Therefore,

$$(4.8) \quad \|I_N F(u)\|_{2,2} \lesssim \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}^{1+\frac{4}{d}}.$$

By (4.7) and (4.8), we obtain

$$(4.9) \quad |(4.4)| \lesssim N^{-\min\{1, \frac{4}{d}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0([0,t])}^{2+\frac{8}{d}}.$$

Collecting (4.2), (4.6), and (4.9), we get

$$|E(I_N u(t)) - E(I_N u_0)| \lesssim N^{-\min\{1, \frac{4}{d}\}s+} (\|\langle \nabla \rangle I_N u_0\|_2^{2+\frac{4}{d}} + \|\langle \nabla \rangle I_N u_0\|_2^{2+\frac{8}{d}}).$$

This proves Lemma 4.2. \square

Next, we use Lemma 4.2 to prove Proposition 4.1.

Let $T < T^*$ and $\Lambda(T)$ and $\Sigma(T)$ defined as in the beginning of this section. By Proposition 3.3, if we take

$$(4.10) \quad \begin{cases} N(T) \gg \Lambda(T)^{\frac{4}{\min\{4,d\}s - (4+d)(1-s) -}} \\ \delta := c_0 \Sigma(T)^{-\frac{2}{s}}, \end{cases}$$

then the solution u satisfies the estimate

$$\|\langle \nabla \rangle I_{N(T)} u\|_{S^0([t, t+\delta])} \lesssim \|\langle \nabla \rangle I_{N(T)} u(t)\|_2 \lesssim \Sigma(T),$$

uniformly in t , provided $[t, t+\delta] \subset [0, T]$. Thus, splitting $[0, T]$ into $O(\frac{T}{\delta})$ subintervals and applying Lemma 4.2 on each of these subintervals, we get

$$(4.11) \quad \begin{aligned} \sup_{t \in [0, T]} |E(I_{N(T)} u(t))| &\lesssim |E(I_{N(T)} u_0)| + \frac{T}{\delta} N(T)^{-\min\{1, \frac{4}{d}\}s+} \Sigma(T)^{2+\frac{4}{d}} \\ &\quad + \frac{T}{\delta} N(T)^{-\min\{1, \frac{4}{d}\}s+} \Sigma(T)^{2+\frac{8}{d}} \\ &\lesssim |E(I_{N(T)} u_0)| + N(T)^{-\min\{1, \frac{4}{d}\}s+} \Sigma(T)^{2+\frac{4}{d}+\frac{2}{s}} \\ &\quad + N(T)^{-\min\{1, \frac{4}{d}\}s+} \Sigma(T)^{2+\frac{8}{d}+\frac{2}{s}}. \end{aligned}$$

Using interpolation, Sobolev embedding, and Lemma 2.4, we estimate

$$\begin{aligned}
 |E(I_{N(T)}u_0)| &\lesssim \|\nabla I_N u_0\|_2^2 + \|I_N u_0\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \\
 &\lesssim N^{2(1-s)} \|u_0\|_{H_x^s}^2 + \|I_N u_0\|_2^{\frac{4}{d}} \|\nabla I_N u_0\|_2^2 \\
 &\lesssim N^{2(1-s)} (\|u_0\|_{H_x^s}^2 + \|u_0\|_{H_x^s}^{2+\frac{4}{d}}) \\
 (4.12) \quad &\lesssim N^{2(1-s)}.
 \end{aligned}$$

Moreover, by Lemma 2.4, we also have

$$(4.13) \quad \Sigma(T) \lesssim N(T)^{1-s} \Lambda(T).$$

Substituting (4.12) and (4.13) into (4.11), we obtain

$$\begin{aligned}
 \sup_{t \in [0, T]} |E(I_{N(T)}u(t))| &\lesssim N(T)^{2(1-s)} + N(T)^{-\min\{1, \frac{4}{d}\}s + (2+\frac{4}{d}+\frac{2}{s})(1-s) + \Lambda(T)^{2+\frac{4}{d}+\frac{2}{s}}} \\
 (4.14) \quad &+ N(T)^{-\min\{1, \frac{4}{d}\}s + (2+\frac{8}{d}+\frac{2}{s})(1-s) + \Lambda(T)^{2+\frac{8}{d}+\frac{2}{s}}}.
 \end{aligned}$$

Optimizing (4.14), we observe that if

$$(4.15) \quad N(T) \sim \Lambda(T)^{\frac{2+\frac{8}{d}+\frac{2}{s}}{\min\{1, \frac{4}{d}\}s - (1-s)(\frac{8}{d}+\frac{2}{s})-}},$$

then $N(T)$ satisfies the assumption (4.10) and moreover,

$$\begin{aligned}
 \sup_{t \in [0, T]} |E(I_{N(T)}u(t))| &\lesssim N(T)^{2(1-s)} \\
 &\lesssim \Lambda(T)^{\frac{2(2+\frac{8}{d}+\frac{2}{s})(1-s)}{\min\{1, \frac{4}{d}\}s - (1-s)(\frac{8}{d}+\frac{2}{s})-}}.
 \end{aligned}$$

Let

$$p(s) := \frac{2(2+\frac{8}{d}+\frac{2}{s})(1-s)}{\min\{1, \frac{4}{d}\}s - (1-s)(\frac{8}{d}+\frac{2}{s})-}.$$

Then, a little work shows that the condition $0 < p(s) < 2$ leads to the restriction

$$s > s_0(d).$$

Thus, for $N(T)$ defined in (4.15) and $s > s_0(d)$, we have $0 < p(s) < 2$ and

$$\sup_{t \in [0, T]} |E(I_{N(T)}u(t))| \lesssim \Lambda(T)^{p(s)}.$$

This proves Proposition 4.1.

5. PROOF OF THEOREM 1.1

In this section, we use Proposition 4.1 together with Lemma 2.11 to prove Theorem 1.1.

We choose a sequence of times $\{t_n\}_{n \geq 1}$, such that $t_n \rightarrow T^*$ as $n \rightarrow \infty$ and

$$\|u(t_n)\|_{H_x^s} = \Lambda(t_n).$$

As the solution u blows up at time T^* , we must have $\Lambda(t_n) \rightarrow \infty$ as $n \rightarrow \infty$.

Set

$$\psi_n(x) := \rho_n^{\frac{d}{2}}(I_{N(t_n)}u)(t_n, \rho_n x),$$

where $N(t_n)$ is given by (4.1) with $T := t_n$ and the parameter ρ_n is given by

$$\rho_n := \frac{\|\nabla Q\|_2}{\|\nabla I_{N(t_n)}u(t_n)\|_2}.$$

By Lemma 2.4 and Corollary 3.2, we get

$$\rho_n \lesssim \frac{1}{\|u(t_n)\|_{H_x^s}} \lesssim (T^* - t_n)^{\frac{s}{2}}.$$

Basic calculations show that $\{\psi_n\}_{n \geq 1}$ is a bounded sequence in H_x^1 . Indeed,

$$(5.1) \quad \begin{aligned} \|\psi_n\|_2 &= \|I_{N(t_n)}u(t_n)\|_2 \leq \|u(t_n)\|_2 = \|u_0\|_2 \\ \|\nabla \psi_n\|_2 &= \rho_n \|I_{N(t_n)}\nabla u(t_n)\|_2 = \|\nabla Q\|_2. \end{aligned}$$

By Proposition 4.1 (with $T = t_n$), we can estimate the energy of ψ_n as follows:

$$E(\psi_n) = \rho_n^2 E(I_{N(t_n)}u(t_n)) \lesssim \rho_n^2 \Lambda(t_n)^{p(s)} \lesssim \|u(t_n)\|_{H_x^s}^{p(s)-2}.$$

Thus, as $p(s) < 2$ for $s > s_0(d)$,

$$E(\psi_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which by the definition of the energy and (5.1) implies

$$(5.2) \quad \|\psi_n\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \rightarrow \frac{d+2}{d} \|\nabla Q\|_2^2 \quad \text{as } n \rightarrow \infty.$$

Applying Lemma 2.11 to the sequence $\{\psi_n\}_{n \geq 1}$ (with $M := \|\nabla Q\|_2$ and $m := (\frac{d+2}{d} \|\nabla Q\|_2^2)^{\frac{d}{2d+4}}$), we derive the existence of a sequence $\{x_n\}_{n \geq 1} \subset \mathbb{R}^d$ and of a function $V \in H^1(\mathbb{R}^d)$ such that $\|V\|_2 \geq \|Q\|_2$ and, up to a subsequence,

$$\psi_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly in } H_x^1 \quad \text{as } n \rightarrow \infty,$$

that is,

$$(5.3) \quad \rho_n^{\frac{d}{2}} (I_{N(t_n)}u)(t_n, \rho_n \cdot + x_n) \rightharpoonup V \quad \text{weakly in } H_x^1 \quad \text{as } n \rightarrow \infty.$$

To prove Theorem 1.1, we have to eliminate the smoothing operator $I_{N(t_n)}$ from (5.3). We do so at the expense of trading the weak convergence in H_x^1 for convergence in the sense of distributions. Indeed, for any $\sigma < s$ we have

$$(5.4) \quad \begin{aligned} &\|\rho_n^{\frac{d}{2}} (u(t_n) - I_{N(t_n)}u(t_n))(\rho_n \cdot + x_n)\|_{\dot{H}_x^\sigma} \\ &= \rho_n^\sigma \|P_{\geq N(t_n)}u(t_n)\|_{\dot{H}_x^\sigma} \\ &\lesssim \rho_n^\sigma N(t_n)^{\sigma-s} \|P_{\geq N(t_n)}u(t_n)\|_{\dot{H}_x^s} \\ &\lesssim \Lambda(t_n)^{-\sigma} \Lambda(t_n)^{\frac{(\sigma-s)p(s)}{2(1-s)}} \|P_{\geq N(t_n)}u(t_n)\|_{H_x^s} \\ &\lesssim \Lambda(t_n)^{1-\sigma+\frac{(\sigma-s)p(s)}{2(1-s)}}. \end{aligned}$$

Plugging the explicit expression for $p(s)$ in the above computation, we find that for

$$\sigma < \tilde{s} := \frac{2d + 8s + s^2 d (2 - \min\{1, \frac{4}{d}\})}{4d + 16s - s^2 (d \min\{1, \frac{4}{d}\} + 8)},$$

the exponent of $\Lambda(t_n)$ in (5.4) is negative. Hence,

$$(5.5) \quad \|\rho_n^{\frac{d}{2}} (u(t_n) - I_{N(t_n)}u(t_n))(\rho_n \cdot + x_n)\|_{H_x^{\tilde{s}-}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Combining (5.3) and (5.5) finishes the proof of Theorem 1.1.

6. PROOF OF THEOREM 1.2

By Theorem 1.1, there exists a blowup profile $V \in H_x^1$, with $\|V\|_2 \geq \|Q\|_2$, and there exist sequences $\{t_n, \rho_n, x_n\}_{n \geq 1} \subset \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}^d$ such that $t_n \rightarrow T^*$,

$$(6.1) \quad \frac{\rho_n}{(T^* - t_n)^{\frac{s}{2}}} \lesssim 1 \quad \text{for all } n \geq 1,$$

and

$$(6.2) \quad \rho_n^{\frac{d}{2}} u(t_n, \rho_n \cdot + x_n) \rightharpoonup V \quad \text{weakly as } n \rightarrow \infty.$$

From (6.2) it follows that for any $R > 0$ we have

$$\liminf_{n \rightarrow \infty} \rho_n^d \int_{|x| \leq R} |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx,$$

which, by a change of variables, yields

$$\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq R\rho_n} |u(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx.$$

As by hypothesis $\frac{(T^* - t_n)^{\frac{s}{2}}}{\alpha(t_n)} \rightarrow 0$ as $n \rightarrow \infty$, (6.1) implies that $\frac{\rho_n}{\alpha(t_n)} \rightarrow 0$ as $n \rightarrow \infty$. Therefore,

$$\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx.$$

Letting $R \rightarrow \infty$, we obtain

$$\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \|V\|_2^2.$$

As $\|V\|_2 \geq \|Q\|_2$, this implies

$$\limsup_{t \rightarrow T^*} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \alpha(t)} |u(t, x)|^2 dx \geq \|Q\|_2^2.$$

As for any fixed time t , the map $y \rightarrow \int_{|x-y| \leq \alpha(t)} |u(t, x)|^2 dx$ is continuous and goes to zero as $y \rightarrow \infty$, there exists $x(t) \in \mathbb{R}^d$ such that

$$\sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \alpha(t)} |u(t, x)|^2 dx = \int_{|x-x(t)| \leq \alpha(t)} |u(t, x)|^2 dx.$$

This finally implies

$$\limsup_{t \rightarrow T^*} \int_{|x-x(t)| \leq \alpha(t)} |u(t, x)|^2 dx \geq \|Q\|_2^2,$$

which proves Theorem 1.2.

7. PROOF OF THEOREM 1.3

In this section, we upgrade Theorem 1.1 to Theorem 1.3 under the additional assumption $\|u_0\|_2 = \|Q\|_2$.

With the notation used in the proof of Theorem 1.1, we have

$$\|\psi_n\|_2 \leq \|u_0\|_2 = \|Q\|_2 \leq \|V\|_2.$$

On the other hand, using the semi-continuity of weak convergence,

$$\|V\|_2 \leq \liminf_{n \rightarrow \infty} \|\psi_n\|_2 \leq \|Q\|_2.$$

Therefore,

$$\|V\|_2 = \|Q\|_2 = \lim_{n \rightarrow \infty} \|\psi_n\|_2.$$

Thus, as $\psi_n(\cdot + x_n) \rightharpoonup V$ weakly in L_x^2 (up to a subsequence which we still denote by $\psi_n(\cdot + x_n)$), we conclude that

$$\psi_n(\cdot + x_n) \rightarrow V \text{ strongly in } L_x^2.$$

Moreover, by the Gagliardo-Nirenberg inequality and the boundedness of $\{\psi_n\}_{n \geq 1}$ in H_x^1 , we have

$$\psi_n(\cdot + x_n) \rightarrow V \text{ in } L_x^{2+\frac{4}{d}}.$$

Combining this with (5.2) and the sharp Gagliardo-Nirenberg inequality, we obtain

$$\|\nabla Q\|_2 \leq \|\nabla V\|_2.$$

By the semi-continuity of weak convergence, we also have

$$\|\nabla V\|_2 \leq \liminf_{n \rightarrow \infty} \|\nabla \psi_n\|_2 = \|\nabla Q\|_2,$$

and so

$$\|\nabla V\|_2 = \|\nabla Q\|_2 = \lim_{n \rightarrow \infty} \|\nabla \psi_n\|_2.$$

Thus, as $\psi_n(\cdot + x_n) \rightharpoonup V$ in H_x^1 , we conclude that

$$\psi_n(\cdot + x_n) \rightarrow V \text{ strongly in } H_x^1.$$

In particular, this implies

$$E(V) = 0.$$

Collecting the properties of V we find

$$V \in H_x^1, \quad \|V\|_2 = \|Q\|_2, \quad \|\nabla V\|_2 = \|\nabla Q\|_2, \quad \text{and} \quad E(V) = 0.$$

The variational characterization of the ground state, [39], implies that

$$V(x) = e^{i\theta} Q(x + x_0)$$

for some $(e^{i\theta}, x_0) \in (S^1 \times \mathbb{R}^d)$. Thus,

$$(7.1) \quad \rho_n^{\frac{d}{2}}(I_{N(t_n)}u)(t_n, \rho_n x + x_n) \rightarrow e^{i\theta} Q(x + x_0) \text{ strongly in } H_x^1 \text{ as } n \rightarrow \infty.$$

Theorem 1.3 follows from (5.5) and (7.1).

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